

were what they are at the present day, it at first sight seems necessary to suppose that there was a rapid change in the temperature in passing from the ice-front towards the Gulf of Mexico. Before we adopt this consideration, however, we must bear in mind the fact that the ice-sheet of the last glacial period probably advanced for a considerable distance south of the perpetual snow-line, in substantially the same way in which an Alpine glacier descends in many cases to a depth of 1000 feet or more below the fields of enduring snow by which it is fed. Accepting the elevation of the continents as they now exist, and allowing  $3^{\circ}$  of temperature for each 1000 feet of altitude, it seems likely that the snow-line just touched the summit of the Carolina mountains and came to the surface of the sea near the southern end of Hudson's Bay. In other words, the protrusion of the ice to the south of this glacial snow-line carried it at a distance of near 1000 miles south of the gathering ground. This supposition, however, is of little value, for the reason that the level of the continent was clearly much disturbed during the glacial period, the surface declining to the northward within the glacial envelope, and probably rising to the southward of the ice-front.

It seems to me most likely that during the occupation of the northern part of the continent by glaciers, the southern portion of the continent was considerably elevated. All the streams which discharge into the ocean south of the former ice-front between New York and the Rio Grande show in their lower parts only moderate accumulations of alluvium which has been deposited since the close of the glacial period. They generally enter bays which appear to be the lower parts of gorges which were formed during the period when the area was more elevated than it is at the present time. These facts make it probable that if the mountains of North Carolina varied in elevation from the present height, they were more elevated than at this day. All the facts are against the supposition that we can explain the absence of glaciers in their highlands by supposing that the summits were lower during the ice period than they now are.

It seems to me we are compelled to suppose that the climate in the mountains of North Carolina, and probably in the great portion of the Apennine section south of the Alps, had during the glacial period a temperature not much if any lower than they have at the present time. As far as it goes, the evidence is thus opposed to the supposition that the glacial period was brought about by a general refrigeration in climate of the continents occupied by the sheet.

Within the basin of the Ohio, especially in the valleys of the Upper Tennessee system of waters, we find certain phenomena which lead us to the conclusion that the rainfall in a recent period, probably contemporaneous with the glacial epoch, was more considerable than at the present day. In many valleys which I have observed in that section the *débris* built into the imperfect alluvial plains is of a much coarser nature than that now brought down by the rivers. The channels bear the aspect of having recently been the seat of more voluminous streams than now occupy them. This evidence, gained from many points in the Southern Appalachians, leads me, independently of the hypothesis I am now suggesting, to the conclusion that during the last glacial epoch the rainfall of this country was much greater than it is at present. At Big Bone Lick in Kentucky, which lies within a few miles of the southern boundary of the ice-sheet, excavations made by me in 1868 show embedded in the deposits formed by the springs an abundant set of herbivorous mammals, including the mastodon and elephant, an extinct species of buffalo, and a musk-ox kindred to our Arctic species but of much larger size, a species of carabou, indistinguishable from our living American forms. The conditions of this deposit led me to suppose that these

animals were probably not more ancient than the glacial period, and that they most likely occupied the surface during the time of abundant rainfall when the marshes were more extensive than at present, a period which if not exactly coincident with the extreme advance of the ice must fall within the glacial epoch.

The abundance of these large Herbivora, the relatively great size of the species, point also to the coincident occurrence of a rather abundant vegetation. If the period indicated by the massive gravels of the torrential streams and the Herbivora of Big Bone Lick be identical, and if this period coincides with the glacial period, as it appears to do, then we may fairly assume that the climatal conditions immediately to the south of the glacial sheet were not those of extreme cold. This evidence has nothing like the sure foundation obtained by the lack of glaciers in the mountains of North Carolina, but as far as it goes it confirms the results of those observations.

It is not my purpose, however, in the present writing to consider the perplexing question as to the cause of glacial climate. I desire only to call attention to the extent to which our glacial streams appear to have advanced, by what we may term forced marches, far to the south of the line of perpetual snow. Although the value of the evidence above noted cannot be determined until the matter has been more carefully brought together and abundantly discussed, the facts seem to me to militate against any hypothesis which seeks to account for the glacial period on the supposition that the climate in the glaciated regions was cooler than at present.

#### THE SUBJECT-MATTER OF EXACT THOUGHT.<sup>1</sup>

WHEN mathematicians, logicians, and other exact thinkers think and reason, they think and reason about something. What is that something? And wherein consists the infinite variety which it presents? Is it a mere assemblage of detached subjects of entirely different natures? Or is it an harmonious whole, admitting of a definition and treatment which, though perfectly general, will yet preserve the essential characteristics of its component parts?

To judge by the usual habit of thought on these questions, we ought apparently to conclude that the former is the correct hypothesis; but that such a conclusion would be wholly erroneous, there can, I think, be no doubt whatever. The prevailing view is due, I believe, to the want of a proper appreciation of the difference between that which is the essential or necessary matter of exact thought, and that which, so far as the processes of reasoning are concerned, is merely the dress in which that necessary matter is clothed. This dress has, of course, a real importance of its own, and the study of it is not to be undervalued. But when we are investigating the subject-matter of exact thought it is not with it that we are concerned: it is not with the case, but with the works of the clock that we have to do; and thus our anxiety should be to get rid of the environments, to treat them as the "disturbing agents" of the experiments of the physicist, as likely to mislead, and therefore to be eliminated with the most scrupulous care.

That such dress exists in the case of every subject that we investigate is obvious enough; much of it is recognized

<sup>1</sup> The object of this paper is to set forth in as simple and non-technical a manner as possible, the principles which were first formulated in my "Memoir on the Theory of Mathematical Form," published in the Philosophical Transactions of the Royal Society for 1886, vol. clxxvii. p. 1, and a "Note" thereon contained in the Proceeding of the Royal Society, vol. xliii. p. 193, which corrects the Memoir in some important particulars. The special applications of the theory principally dwelt on here are considered in detail in a recent paper by me, "On the Relation between the Geometrical Theory of Points and the Logical Theory of Classes," contained in the Proceedings of the London Mathematical Society, vol. xxi. p. 147. The mode of treatment of the whole subject adopted in the present communication must however be regarded as entirely new.—A. B. K.

without an effort. Thus the geometrician must draw the lines of his diagrams of some colour, but never dreams of supposing that the particular colour affects the real matter he is investigating; and the logician, bidding us fix our attention on the relations of implication, contradiction, &c., of the passages he selects for analysis, properly rejects the poetic beauties or philosophic truths they may enshrine as matters with which he is not for the moment concerned. But though there is a great deal of this accidental environment which is readily seen to be such, and is therefore rejected without difficulty, there remains much which, unless attention is expressly directed to its discovery and rejection, is likely to be regarded as the kernel itself, and not as mere husk. In innumerable instances this separation of the essential from the non-essential has, I am satisfied, not yet been effected. That this is so is, to a very considerable extent, due to the practice, no doubt for some purposes a necessary one, of dividing the study of the subject-matter of exact thought into different sciences, such as logic and mathematics, each with its own literature and students, the latter generally misinformed as to the real nature of those subjects which are not their special study. By this divorce of studies the student of either has the dress in which his particular subject is clothed so invariably associated with its essential elements that he fails to regard it as a dress at all, and looks upon it as a part of the naked body itself. No wonder then that such attempts as have been made to define the nature of the subject-matter of certain sciences are in general vague and metaphysical, rather than exact and mathematical; or that there are those who are content to say that logic is the science of "quality," mathematics of "quantity," and therefore their domains are, and should be kept, distinct. Such persons would no doubt be surprised to learn that mathematics is no more the science of quantity only than physiology is the science of the arm or leg; and that the algebra which expresses the relations of quantity is but a drop in the ocean of algebras which the mathematician can point to; another drop, be it observed, being the algebra which expresses the logical relations of classes or propositions to each other.

The essential elements of the subject-matter of exact thought are in reality of an extremely simple character; and, though they exhibit infinite variety, that variety is due to simple and easily defined causes. There is nothing vague or metaphysical about them; but, even when mere figments of the brain, they are precise and definite, with parts and properties which can be analyzed and catalogued, just as much as if they were the elements of a chemical compound, the wheels of a watch, or the organs of a vital structure. Let me try to show that this is so.

I will begin by considering and comparing the essential matter of two "branches of science," which will, I think, be regarded by most persons as of quite different characters, and as very properly relegated to separate and distinct treatises. I refer to the geometrical theory of points, and the logical theory of statements. The investigation will, I hope, fully prepare the way for an acceptance of the general definition of the subject-matter of exact thought which will follow.

I take points first. The geometrician deals, of course, with many things besides points, viz. lines, planes, curves, surfaces, &c.; but points may be considered alone, and it conduces to much greater clearness of ideas to confine ourselves to one species of entity, instead of introducing several into our field of view. I deal therefore with points only. And first let us consider individual points. Euclid says that they have neither parts nor magnitude; but these facts, however true and interesting, are, so far as the exact thought of the geometrician is concerned, beside the mark, and introduce considerations which tend to divert our attention from that which is really essential. When these immaterial considerations

are swept on one side, all that we can say about points as individual entities is that they are all exactly like each other, are undistinguished from each other; no remark can be made of one which is not equally true of every other. But there are innumerable other entities about which the same thing can be said; and yet these do not possess those properties which characterize points. To say that points have *positions* is no explanation: it merely leads us to consider positions, and these are really the same things as points. Thus far, then, we have not obtained much insight into the essential characteristics of a system of points.

Let us go a step further and consider *pairs* of points. But even this will help us but little, for pairs of points are geometrically speaking, undistinguished from each other, just as single points are. No remark can be made about any one pair of points which is not equally applicable to every other pair. It is true, no doubt, that in some pairs the points are further apart than in others; but in considering distances we are not considering properties of individual pairs of points, but the relations of certain pairs to each other; and when we consider the relation of one pair of points,  $a, b$ , to another pair,  $c, d$ , we do not deal with peculiarities of the individual pairs  $a, b$ , and  $c, d$ , but with a peculiarity of the tetrad of points,  $a, b, c, d$ . To put the matter in another way, the geometrical properties of a diagram do not depend upon its size; whether it be enlarged or diminished they remain the same.

We are still, then, as much in the dark as ever as to what there is about a system of points, which gives to it the properties which are discussed by the geometrician; and we must advance yet another step, and consider *triads* of points. And here at last light breaks in upon us. Unlike single points and pairs of points, triads of points are not exactly like each other, and remarks may be made of certain triads of points, which, though true of some others, are not true of all. In the first place, there is the great division of triads of points into *collinear* triads, i.e. triads consisting of three points through which a straight line could be drawn, and *non-collinear* triads through which no such line could pass. This division is of fundamental importance, and the consideration of it introduces us to the whole of those geometrical properties which, for reasons into which I need not now enter, are termed "projective." But besides this division into collinear and non-collinear triads, there are apparently subdivisions of the triads of each sort, arising from the differences which may exist in the relative distances of the three points in a triad from each other—in the *shape* of the triangle which they form. The geometrical properties which depend on such subdivisions are known as "metrical" properties. We need not, however, separately consider these subdivisions; for, thanks to the researches of Poncelet and Cayley, we now know that the consideration of "projective" properties comprehends the study of those which are "metrical"; and thus a complete discussion of the results which flow from the division of triads of points into collinear and non-collinear triads, would include a full examination of those which arise from metrical subdivisions. In fact, when we consider the metrical properties of triads of points we are not considering triads at all, but collections of a larger number of points, arrived at by adding to the triads certain other points technically known as "the absolute." If we consider the triads alone and apart from "the absolute," i.e. if we consider their projective properties, we have only two sorts of triads to deal with, viz. collinear and non-collinear. All the triads of each sort are precisely alike in properties, nothing can be said of one of the triads of one sort which is not true of each of the others of that sort; but every collinear triad differs radically from every non-collinear triad.

All, therefore, that we need now concern ourselves with is the division of triads of points into collinear and non-collinear triads. Now, though the distinction between

collinear and non-collinear triads is of fundamental importance, it is not the collinearity and non-collinearity in themselves which are the subjects of consideration by the geometrician. The fact that in a collinear triad the three points are characterized by a symmetry or straightness which is not found in triads of the other species is, so far as the geometrician is concerned, a matter of indifference; it is a piece of accidental clothing which is not an essential part of the subject-matter with which he is dealing. He is concerned with the fact that triads of the one species differ radically from those of the other; but not with the fact that they differ in respect of straightness. With what, then, is he concerned beyond the mere fact that they differ? The answer is, that it is with the way in which the triads of the two species are distributed through the whole system of points; and with the fact that this distribution is not a random one, but is regulated by definite laws, so that if we are told that certain triads are collinear or non-collinear it necessarily follows that certain others are also collinear and non-collinear respectively.

The laws regulating this distribution may be stated as follows:—

LAW I.—If the two triads  $a, b, p$ , and  $c, d, p$  are collinear triads, there exists a point  $q$  such that the two triads  $a, c, q$  and  $b, d, q$  are collinear triads.

LAW II.—If the two triads  $a, b, c$  and  $b, c, d$  are collinear triads, so also are the triads  $a, c, d$  and  $a, b, d$ .

LAW III.—No point is absent from the system whose presence is consistent with the foregoing laws.

The distribution of the triads of the two species through the system in accordance with these laws completely determines it as one possessing all those properties which are usually studied by the geometrician. I insert the limitation “usually,” because there are certain matters sometimes considered which require other circumstances than those previously referred to be taken into account. To these I shall return later.

It necessarily, of course, follows from this definite distribution of the distinguished and undistinguished triads that there exists a similar regularity of distribution of distinguished and undistinguished collections of any number of points. But, in addition to this, there also exists a like regular distribution of what may be termed “aspects” of those collections, about which I must say something.

By an “aspect” of anything in the ordinary sense of the word, we mean the appearance which it presents under conditions. If the conditions are altered, we have different aspects of the thing. Thus, a thing presents different aspects when viewed from different standpoints, or when placed among different surrounding circumstances. The conditions may be purely mental. Thus, when we consider the relation of one of a collection of objects to the other objects of the collection, we mentally attach a different degree of prominence to the former object from that which we assign to the latter objects, and we have a particular aspect of the whole collection in view. An aspect of a collection of entities will also be obtained if we regard the entities of the collection as taken in a particular order. It is with aspects of this latter description that we are here concerned. It must not, however, be supposed that the *order*, in the sense of *succession*, is of any importance so far as we are here concerned. All that is material is the fact that the entities are conceived of as taken first, second, third, &c., and that to be taken first is different from being taken second, and so on, the nature of the difference being immaterial; so that an aspect of the sort considered would equally well be obtained by conceiving that each entity of the collection is marked with a different mark of any sort. It is, however, convenient for my purpose to consider the aspects obtained by supposing the entities of the collection to be taken in a particular order; among other reasons, because such aspects admit of being represented

in an extremely simple manner; viz. if  $a, b, c, d, \dots$  represent the entities of any collection of entities, we may represent the various aspects of that collection by arranging those letters in rows, thus  $abcd, \dots$ , or  $bdc, \dots$ . Aspects may be different, and yet be undistinguished from each other. Thus, by looking at a sphere from different points of view we obtain different aspects of it, but one of these may be exactly like another. So in the case of an aspect of a collection of any number of points; for example, to take the simplest case, of a pair of points,  $a, b$ . Here the two aspects  $ab$  and  $ba$  are different aspects of the pair, but they are undistinguishable from each other; nothing can be said of a pair of points taken in one order which is not true of the pair taken in the reverse order. In other words, a pair of points is *symmetrical*. On the other hand, if a collection of points is *unsymmetrical*, e.g. if it consists of a collinear triad,  $a, b, c$ , and one point,  $d$ , which is not collinear with  $a, b, c$ , then we can always find two aspects of the collection which are distinguished from each other, e.g. in the given case the aspects  $abcd$  and  $abdc$  are distinguished from each other.

The consideration of these aspects of collections of points is of much use in the discussion of the properties of a system of points. To take one simple example. Suppose we were told that the collection of four points,  $p, q, r, s$ , is undistinguished from the collection  $a, b, c, d$ , which we have just been considering. Then, though we know that three of the points  $p, q, r, s$  compose a collinear triad, we do not know which three; but if we are told that the aspects  $abcd$  and  $prsq$  are undistinguished from each other, then we know that, since  $a, b, c$  is a collinear triad,  $p, r, s$  is one also.

Every collection of points has, then, a number of aspects, arrived at by taking the points composing it in every possible order; and of these aspects of the various collections which compose a system of points, some are distinguished from each other, and some undistinguished; and these distinguished and undistinguished aspects are regularly distributed through the system, just as the distinguished and undistinguished collections are distributed. This regular distribution of the aspects is determined by the distribution of the collinear and non-collinear triads of points in accordance with the laws which I have given, and on that distribution alone.

Owing to this regular distribution of the distinguished and undistinguished collections, and aspects of collections, which compose a system of points, that system possesses a specific character or “form,” as it may be termed. It is to the possession of this “form” that the system owes its various properties as studied by the geometrician, and it is this “form,” and that only, which is the real subject-matter of his exact thought in the study of a system of points.

I do not propose to show here how the various properties of a system of points necessarily follow from the laws which I have given as defining its “form,” or how it is that such a system contains sets of points having the characteristic properties of such as lie on straight lines, curves, surfaces, &c. These are all matters of detail involving no new questions of principle, and can be worked out without difficulty by anyone who has a knowledge of modern projective geometry. It is sufficient for my purpose to have indicated the position which “form” occupies in the geometrical theory of points.

Before, however, I pass away from the consideration of points, I must call attention to one or two matters with regard to them, which, as I have already indicated, may on occasion have to be considered by geometricians, and are not covered by the foregoing treatment.

I have hitherto assumed that the coincidence of points implies their identity, and in most geometrical investigations this is taken to be so. We may, however, regard coincidence as amounting merely to equivalence, i.e. we may regard coincident points as such that each bears

precisely the same relation to all other points.<sup>1</sup> Where a system of points thus includes equivalent points, pairs of points are of two sorts, viz. we have equivalent pairs and non-equivalent pairs. Equivalent pairs are distributed through the whole system of points in accordance with the following law:—

If each of the pairs  $a, b$  and  $a, c$  is an equivalent pair, then the pair  $b, c$  is an equivalent pair.

Another matter to which reference should be made is this. The laws which I have given as defining the "form" of a system of points define that form by specifying the mode of distribution of the collinear and non-collinear triads, i.e. of the various *collections* of three points to be found in the system. Nothing is said about the *aspects* of those triads, and thus we do not know, in the case of any collinear triad, which is the mean point of the triad, and which are the extremes—which point lies between the other two. In general it is not necessary that we should know this; in a vast number of geometrical investigations it is wholly immaterial. In certain cases, however, the matter is of importance; and, as its consideration brings out a very interesting and important fact as to the relation between points and statements, I must not pass it over. Here the laws which define the distribution of the collinear triads must not only specify what triads are collinear, but also which is the mean point and which the extremes in each. It must, however, be noticed that the fact that in a collinear triad one point "lies between" the other two is not a material circumstance; all that is of importance is that the one point bears a different relation to the triad from that which is borne by the other two points. It is convenient to denote a collinear triad in which  $b$  is the mean point, and  $a$  and  $c$  the extremes, by the symbol  $ac.b$ . The laws of distribution of the collinear triads may then be stated thus:—

LAW I.—If we have the collinear triads  $ap.b$  and  $cp.d$ , a point  $q$  exists such that we have the collinear triads  $ad.q$  and  $bc.q$ .

LAW II.—If we have the collinear triads  $ab.p$  and  $cd.p$ , a point  $q$  exists such that we have the collinear triads  $aq.d$  and  $bc.q$ .

LAW III.—If we have the collinear triad  $ab.c$ , and  $a$  and  $b$  are equivalent points, all the three points,  $a, b, c$ , are equivalent.

LAW IV.—If  $a$  and  $b$  are equivalent, we have the collinear triads  $ac.b$  and  $bc.a$ , whatever point  $c$  may be.

LAW V.—If the triads  $a, b, c$  and  $b, c, d$  are both collinear triads, so also are both the triads  $a, c, d$  and  $a, b, d$ .

LAW VI.—No point is absent from the system whose presence is consistent with the foregoing laws.

I shall have occasion again to refer to these laws, but, for the present, I pass away from the consideration of the geometrical theory of points, and proceed to discuss the logical theory of statements.

In place of points, we have now to deal with statements. Most persons would, I think, say that our new subject-matter is something altogether different from that with which we have hitherto been dealing, and would demur to my observation that, as subjects of exact thought, statements are just the mere entities that points are. "Statements," they would say, "are complex structures; some very complex, consisting, in fact, of a number of other simpler statements combined together by the use of the conjunctions 'and' and 'or'; and, of the simpler statements, even the most simple comprise parts—'terms,' &c.—which cannot be overlooked. It would be

absurd to say that all these are to be ignored, and a statement regarded as if it were a mere entity such as a point is."

In answer to such objections, I would point out, on the one hand, that a statement is not other than a mere entity because it happens to be expressed as a combination of other statements, any more than a number is other than a mere number when it is expressed as the sum or product of other numbers. Nor, on the other hand, do we, by regarding a so-called "complex" statement as a mere entity, ignore the other statements in terms of which it is expressed, any more than we ignore certain numbers when we regard the number which is their sum or product as a single number, and not as a sum or product of two or more numbers. Those other statements will not be ignored, but will be regarded and treated each as a distinct entity. Every statement when considered alone is regarded as a single entity. When it is taken in conjunction with others, we see that it bears to them certain relations which we call "inconsistency," "implication," &c.; and, by virtue of the existence of these relations, it may be expressed in terms of those other statements, just as one number may be expressed as the sum or product of other numbers to which it is related.

As regards the "terms," &c., which compose statements, I remark that we shall here be concerned only with the relations of statements to each other, and not with the relations that they bear to "terms" or other things. The relations which statements bear to each other may, of course, be considered and expressed by dealing with the terms, &c., which compose them; and equally the relations of points to each other might be expressed by reference to the straight lines, curves, surfaces, &c., which pass through them: but, just as points have relations to each other which may be considered without reference to other geometrical conceptions, so statements and their mutual relations may, and will here be, discussed without any regard to terms.

But a further objection may be raised to the notion that statements as the subjects of exact thought are mere entities, such as points are, which must be considered and dealt with; and that is, that it leaves no place for those conceptions of the truth and falsity of statements which seem to be of the essence of the logical theory: ideas of truth and falsehood can hardly be associated with mere entities. In order to dispose of this objection, let us consider what the logical theory of statements is, as it is usually understood. Statements, with certain special exceptions to be presently referred to, are conceived of by the logician as admitting of being regarded at will as either true or false. This liberty, which we have to regard individual statements as either true or false, does not extend to all pairs, triads, &c., of those statements; for, in general, if certain statements are regarded as true, there are others which must be regarded as true also, and others which must be regarded as false. Similarly, if certain statements are regarded as false, there are others the truth or falsity of which is thereby determined. Our liberty, then, in this respect is subject to certain restrictions. It is to these restrictions that statements owe those mutual relations which it is the object of the logician to investigate and define.

I have said that there are certain special exceptions to the rule that individual statements are conceived of as admitting of being regarded at will as either true or false. The restriction, in fact, which exists in the case of pairs, &c., of statements, extends equally to the case of individual statements, for there are some which cannot be regarded at will as either true or false, but must be regarded some as always true, and others as always false. These statements are called *truisms* and *falsisms* respectively. Logically speaking, all truisms are equivalent: each bears precisely the same logical relation to every other state-

<sup>1</sup> Observe here the difference between undistinguishableness and equivalence. In order that two entities may be undistinguished, it is sufficient that the relation which one bears to any collection of entities may be borne by the other to a collection which is undistinguished from the former collection. But, in order that two entities may be equivalent, it is necessary that the relation which one bears to any collection of entities should be borne by the other to the same collection, and not merely to one which is undistinguished from it.

ment or body of statements; and this is equally the case with falsisms. The introduction of truisms and falsisms into our field of view brings us to the root of the difficulty about the truth and falsehood of statements, and enables us to dispose of it. For to regard a statement as true is merely to ignore the difference between it and a truism—to regard it as equivalent to a truism; and the conception of the truth of a statement is thus simply one as to the *equivalence* of two statements one of which is a truism. In the same way the conception of the falsity of a statement is one as to the equivalence of two statements one of which is a falsism; and generally the logical relation between statements which is expressed by saying that if certain statements are true or false certain others are also some true and some false, is one which may equally well be expressed by saying that if certain statements, one of which is a truism, or a falsism, are equivalent, so also are certain others, one of which is a truism, or a falsism. The truth of a truism and the falsity of a falsism are not matters with which we are here concerned at all, any more than we are with the elegance or conciseness of the language in which they are couched; the question is one merely as to the equivalence of statements, and the relations expressed are such as may and do exist between statements no one of which is a truism or falsism. In fact, truisms and falsisms, as regards the logical relations which they bear to other statements, differ in no material respects from any other statements; and indeed all statements are, so far as their logical relations are concerned, undistinguished from each other; for, whatever relation a statement bears to any body of statements, that relation is also borne by every other statement to some other body of statements.

How comes it, then, it may be asked, that truisms and falsisms unquestionably do appear to bear exceptional relations to other statements? The reply is simple. We are accustomed to consider statements with reference to the relations which they bear to truth and falsehood, *i.e.* to truisms and falsisms, and the verbal shape which they assume in general involves such a reference. In fact, as we shall presently see, whenever the words “and” or “or” are used, there is such a reference involved. Statements which thus involve a reference, whether express or implied, to the relations which they bear to truisms or falsisms naturally seem to bear exceptional relations to the latter; though in fact, logically speaking, they bear no such exceptional relations.

Statements, then, as subjects of the exact thought of the logician, compose a system of entities which are undistinguished from each other. Let us proceed, as in the case of points, to consider pairs of these entities. At first sight it may seem that pairs of statements are of many different sorts; for two statements may be equivalent, inconsistent, contradictory, one of them may imply the other, and so on. But, as regards certain of these relations, a little examination in the light of the preceding observations will make it clear that they are not really relations between *two* statements at all, but between *three*, one of which is a truism or a falsism. Thus two “inconsistent” statements are such that they and a truism cannot all three be regarded as simultaneously equivalent; and similarly in other cases, which will be considered when we come to deal with triads of statements. There are, in fact, but three species of pairs, *viz.* *equivalent* pairs, *contradictory* pairs, and *simple* pairs.

An *equivalent* pair consists of two statements which are such that, whatever logical relation one of them bears to any body of statements, that same relation is borne by the other to the same body. Equivalent pairs are distributed among the whole body of statements in accordance with the following law:—

If each of the pairs  $a, b$  and  $a, c$  is an equivalent pair, then the pair  $b, c$  is also an equivalent pair.

A *contradictory* pair, as usually defined, consists of two

statements which cannot both be regarded as true or both as false, and the relation considered would therefore seem to be one between *four* statements, and not *two*. This is not, however, really the case. The relation may no doubt be thus defined by reference to the two additional statements—a truism and a falsism; but it may also be fully defined without reference to any such additional statements: *viz.* two statements are contradictory if they cannot be regarded as equivalent without ignoring all logical relations. Since two contradictory statements cannot be equivalent to each other, it of course necessarily follows that they cannot both be equivalent to a third statement, whether it be a truism or a falsism, or any other statement. Two statements which are a contradictory pair may be said to be *obverses* of each other. A truism and a falsism are obverses of each other.

The following law holds with regard to contradictory and equivalent pairs, *viz.* :—If  $a, b$  and  $a, c$  are both contradictory pairs, the pair  $b, c$  is an equivalent pair. As however this law is a necessary consequence of certain laws which we shall have presently to consider, I do not include it among the fundamental laws which define the properties of a system of statements.

The third sort of pair, *viz.* a *simple* pair, is one which is neither an equivalent pair nor a contradictory pair: *i.e.* it consists of two statements which are neither necessarily equivalent nor necessarily non-equivalent, but may at will be regarded either as equivalent or not.

The division of pairs of statements into the three species, and the distribution of the pairs of the different species in accordance with the foregoing laws is not enough to determine the properties of a system of statements; and we must, as in the case of points, go on to consider triads of statements. There are of two sorts, which, for reasons that will presently appear, I term “linear” and “non-linear” respectively. There are other subdivisions of the triads into different sorts, but the division into linear and non-linear triads determines the other subdivisions. As in the case of a collinear triad of points, a linear triad of statements consists of two statements which may be called the “extremes,” and one which may be called the “mean.” It is such that if the two extreme statements are regarded as equivalent the mean must also be regarded as equivalent to them, and in this respect also it resembles a collinear triad of points. Any three statements which are thus related compose a linear triad, and any three which are not so related compose a non-linear triad. I shall employ the same symbol to denote a linear triad of statements that I employed in the case of a collinear triad of points; *viz.* a linear triad in which  $a, b$  are the extremes and  $c$  is the mean statement will be denoted by  $abc$ .

These linear triads are not scattered at random through the whole body of statements, but are distributed in accordance with the following laws:—

LAW I.—If we have the linear triads  $apb$  and  $cpd$ , a statement  $q$  exists such that we have the linear triads  $adq$  and  $bcq$ .

LAW II.—If we have the linear triads  $abp$  and  $cpd$ , a statement  $q$  exists such that we have the linear triads  $aqd$  and  $bcq$ .

LAW III.—If we have the linear triad  $abc$ , and  $a$  and  $b$  are equivalent statements, all the three statements  $a, b, c$  are equivalent.

LAW IV.—If  $a$  and  $b$  are equivalent, we have the linear triads  $ac.b$  and  $bc.a$ , whatever statement  $c$  may be.

LAW V.—No statement is absent from the system whose presence is consistent with the foregoing laws.

The distribution through a system of statements of the triads of the two species in accordance with the foregoing laws completely defines the system as one possessing all those properties of statements which are really under consideration by the logician when studying the relations of statements to each other. The fact that the extreme state-

ments of a linear triad cannot be regarded as equivalent without also regarding the mean statement as equivalent to them is a necessary consequence of Law III.; and thus all that is essential in a system of statements, so far as the exact thought of the formal logician is concerned, is that it is a system of entities, pairs and triads of which are of different sorts, and are distributed through the system in accordance with the specified laws.

This uniform distribution of pairs and triads of statements involves of course a similar regularity of distribution of distinguished and undistinguished collections of larger numbers of statements, and also of the aspects of those collections; and consequently a system of statements possesses "form," and owes its properties to the possession of this "form," in precisely the same way as a system of points does. Thus in the case of the logical theory of statements, as in that of the geometrical theory of points, it is "form" as here defined which is the real subject-matter of exact thought.

A remarkable circumstance connected with the laws defining the "form" of a system of points, and that of a system of statements, will no doubt have already been noticed. If we exclude Law V. of the former set of laws, the two sets of laws are the same; and it is thus on this Law V., and that only, that the differences between the properties of the two systems depend. This Law V. is that which expresses the fact that two straight lines can only cut once; so that, if in geometry this restriction were removed, the study of points would in all that is essential be the same thing as the study of statements. I cannot pursue this very interesting fact any further here; but it will now be understood why the expression "linear" has been used with reference to certain triads of statements.

As the views here put forward as to the true nature of the subject-matter of the exact thought of the logician in his consideration of the mutual relations of statements are somewhat novel, I cannot well pass on with the observation that the rest is mere matter of detail, but must briefly allude to one or two matters of importance. And first let us consider those relations which are usually considered as relations between *two* statements, but are, as I have already said, really relations between *three* statements, one of which is a truism or a falsism.

In a linear triad, let the mean statement be a falsism, then the extremes cannot both be regarded as true. For to regard them both as true is to regard them as both equivalent to a truism, *i.e.* as equivalent to each other. But if they are equivalent they must be equivalent to the mean statement, which is a falsism, *i.e.* they must be false and not true. The two extremes are here, therefore, *contrary or inconsistent*.

If we take the mean statement of a linear triad to be a truism, then the extremes cannot both be regarded as false, they are *subcontrary*.

If we take one of the extremes to be a truism, then, if the other extreme is regarded as true, the mean must also be so regarded. Here the two latter statements are *subalterns*, the extreme being the *subalternant*, and the mean the *subalternate*. In common parlance the former statement "implies" the latter.

Next let me point to one instance in which certain relations of statements to each other involve others. If we have the linear triads  $ab.c$  and  $bc.d$ , it can be shown to be an immediate consequence of the laws which I have given as defining the "form" of a system of statements, that we have also  $ab.d$  and  $ad.c$ . Taking, then,  $b$  to be a truism, this becomes:—If the statement  $a$  implies the statement  $c$ , and the statement  $c$  implies the statement  $d$ , then the statement  $a$  implies the statement  $d$ , and  $ad.c$  is a linear triad. Observe here that the last part of the conclusion is not usually pointed out, because the fundamental character of the linear triad has not been noticed.

I proceed finally to consider the use of the words

"and" and "or," and the logical relation of statements such as " $a$  and  $b$ ," " $a$  or  $b$ ," to the statements  $a, b$ . If  $a, b, c$  be any three statements whatever, there exists a statement  $x$ , which is such that the three triads

$$ab.x, bc.x, ca.x$$

are all linear triads. This statement  $x$  is uniquely related to the triad  $a, b, c$ ; that is, it is a necessary consequence of the laws defining the system that, if any other statement,  $y$ , is such that the triads

$$ab.y, bc.y, ca.y$$

are all linear triads, then the statements  $x$  and  $y$  are equivalent. This statement,  $x$ , I term the *symmetrical resultant* of the triad  $a, b, c$ .

Now let  $t$  be a truism, and  $f$  a falsism. Then the symmetrical resultant of the triad  $a, b, f$  is the statement usually written

$$"a \text{ and } b,"$$

and the symmetrical resultant of the triad  $a, b, t$  is the statement usually written

$$"a \text{ or } b,"$$

where, however, the "or" would be more properly written " $\frac{\text{and}}{\text{or}}$ ," as in mercantile documents.

In the general case, in which we take any three statements,  $a, b, c$ , without taking one of them to be necessarily either a truism or a falsism, the symmetrical resultant will be

$$"a \text{ and } b, \text{ or } b \text{ and } c, \text{ or } c \text{ and } a,"$$

or

$$"a \text{ or } b, \text{ and } b \text{ or } c, \text{ and } c \text{ or } a,"$$

the two statements being equivalent. These statements have tacit reference to a truism and a falsism, as they are expressed in terms of the statements " $a$  and  $b$ ," " $a$  or  $b$ ," &c., which are, as we have seen, symmetrical resultants of the triads  $a, b, t$ , and  $a, b, f$ , &c. But no truism or falsism is really involved in arriving at the symmetrical resultant of  $a, b, c$ ; it is a function of  $a, b$ , and  $c$ , only. As, however, there is no simpler verbal expression for the symmetrical resultant in vogue, we are obliged to have recourse to one which expresses it by tacit reference to statements which are not really involved; just as, in the algebraic treatment of geometry, the relations of points to each other are expressed by means of an algebra which has tacit reference to an "origin" and "axes," which are not really involved in the relations which are represented.

I now pass away from the special consideration of statements.

The principles which I have enunciated in the case of points and statements may readily be shown to extend to entities of other descriptions. Thus, straight lines, curves, surfaces, &c., as individuals are mere entities, just as points are. Some are undistinguished from each other, and some are distinguished; thus one plane is exactly like another, but is distinguished from a straight line. These various entities owe their properties to the fact that they bear various relations to each other, and to points. The relations which they bear to each other are defined when we know those which they bear to points; and the relations which they bear to points are due simply to the fact that some points lie on them, and others do not. That a point does or does not "lie on" a curve is not, however, of itself a material circumstance; it is accidental clothing. All that is material to the geometrician is, that the pair of entities consisting of a curve and a point which lies on it is distinguished from the pair which consists of that curve and a point which does not lie on it, and that the two sorts of pairs are distributed through the whole system of points and curves in a definite way. Though the relations of these other geometrical

entities to each other may be arrived at by considering the relations which they bear to points, we may of course consider their relations without reference to points. Thus the system which consists wholly of straight lines is one composed of entities which are undistinguished from each other, and are such that pairs of those entities are of two sorts, viz. two straight lines cut or do not cut each other; and these two sorts of pairs are distributed through the system in accordance with definite laws.

Such considerations as these apply, not merely to straight lines, curves, surfaces, &c., but to all other entities which are dealt with by the geometrician. Vectors, quaternions, matrices, and even algebras, are as individuals all mere entities, and owe their properties to the fact that they bear certain relations to the entities previously referred to, and to each other; and all that is essential in these relations depends merely on the fact that certain individuals, pairs, triads, &c., and aspects of these, are distinguished from each other, and certain undistinguished, and have a specific distribution. Here therefore, also, it is "form" in the sense in which I have defined that word, and "form" only, which is under consideration.

In the same way terms, classes, syllogisms, and other logical conceptions are all mere entities; and all that is essential in their properties, so far as the exact thought of the formal logician is concerned, depends upon the fact that they compose systems possessing "form," i.e. a definite distribution of distinguished and undistinguished collections of entities, and of aspects of those collections.

A system of entities under consideration in any investigation will usually comprise entities of many sorts, some of which are taken into account merely for the purpose of facilitating the study of the properties of others. Algebras, and operations such as quaternions, are examples of entities thus added by the geometrician to a system of points, lines, curves, &c., in order to aid in arriving at the properties of the latter. By thus adding "auxiliary" entities to a system, we may also greatly simplify the definition of its "form"; so that a system which would otherwise be defined only by reference to the distribution of aspects, or of collections of a large number of entities, can be defined by definitions which refer merely to the distribution of collections of a small number of entities, and make no allusion to aspects. In fact, it may be shown that, by the addition of suitable entities, the "form" of any system of entities whatever may be defined by specifying merely that certain *individual* entities, and certain *pairs* of those entities, are like and certain unlike.

The light which is thrown by the foregoing investigations will, I hope, be sufficient to insure the appreciation, if not the acceptance, of the following general definition of the subject-matter of exact thought:—

Whatever may be the true nature of things, and of the conceptions which we have of them (into which points we are not here concerned to inquire), in the operations of exact thought they are dealt with as a number of separate entities.

Every entity is distinguished from certain entities, and (unless unique) is undistinguished from others. In like manner every collection of entities is distinguished from certain collections of entities, and (unless unique) is undistinguished from others; and every aspect of a collection of entities is distinguished from certain aspects of collections, and (unless unique) is undistinguished from others.

Every system of entities has a definite "form," due (1) to the number of its component entities, and (2) to the way in which the distinguished and undistinguished entities, collections of entities, and aspects of collections of entities, are distributed through the system.

The peculiarities and properties of a system of entities

depend, so far as the processes of exact thought are concerned, upon the particular "form" it assumes, and are independent of anything else.

It may seem in some cases that other considerations are involved besides "form"; but it will be found on investigation that the introduction of such considerations involves also the introduction of fresh entities, and then we have merely to consider the "form" of the enlarged system.

If the definition of the subject-matter of exact thought which I have thus ventured to formulate be an accurate one, it will be obvious that there are divisions at present maintained between different branches of exact science which must be regarded as unnecessary and arbitrary; the only differences which should be treated as material being such as are due to differences of "form." These differences are certainly not such as to justify the isolation and separate study of individual systems, without any attempt to fix the position which each holds in the general body of possible systems, or to ascertain the exact points in which each resembles or differs from other systems of the whole body. The scientific study of each system must involve that of the properties common to all, of the general laws regulating the distribution of distinguished and undistinguished collections and aspects of collections in systems of any "form," and of the possibilities of variety in their "forms."

A. B. KEMPE.

#### NOTES.

THE retirement of Sir Gabriel Stokes from the Presidency of the Royal Society is to be taken as an occasion for the expression of the Fellows' high appreciation of his services. He has held office, either as one of the secretaries or as president, for thirty-six consecutive years; and it is unnecessary to say how much the Royal Society has benefited by his labours during that long period. It is thought that the most suitable way of marking the present occasion would be to obtain a good portrait of Sir Gabriel for the gallery of the Society; and an influential committee has been appointed to make the necessary arrangements.

AT a meeting of the City side of the Gresham Committee at Mercers' Hall on Monday, Mr. Karl Pearson, Professor of Mechanics and Applied Mathematics in University College, was elected Gresham Lecturer in Geometry, in succession to the Dean of Exeter.

DR. J. JAGOR, the eminent ethnographical traveller, intends to make a scientific tour in British India. Remembering his researches forty years ago, many men of science in Berlin take much interest in his present plans.

THE German Colonial Society have forwarded a number of books on tropical plants to Emin Pasha to further his scientific researches, as he lately complained (in a letter to Prof. Schweinfurth) of his want of works of reference. Prof. Noack has lately received a letter from the Pasha, dated Tabora, the middle of August. He then intended to leave for Uramba in four or five days, on his way to Lake Tanganyika.

AT a meeting of the Royal Botanic Society on Saturday, the Secretary answered various questions as to the destructive action of fogs on plants. He said it was most felt by those tropical plants in the Society's houses of which the natural habitat was one exposed to sunshine. Plants growing in forests or under tree shade did not so directly feel the want of light; but then, again, a London or town fog not only shaded the plants, but contained smoke, sulphur, and other deleterious agents, which were perhaps as deadly to vegetable vitality as absence of light. Soft, tender-leaved plants, and aquatics, such as the *Victoria regia*, suffered more from fog than any class of plants he knew.